EULERIAN FORMULATION FOR LARGE DISPLACEMENT ANALYSIS OF SPACE FRAMES

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REPRINT SUMMARY

This paper presents a new procedure for the nonlinear analysis of space frames. The procedure is based on an incremental application of rotations in 3D space, coupled with an improved rotational transformation matrix, element-based vectors, and an Eulerian quartic formulation.

1. ABSTRACT

In this paper, a new general procedure is presented for modelling the effects of large displacements on the response of space frames subjected to conservative loading. An incremental definition of rotations is adopted based on an improved rotational transformation matrix, and a convected (Eulerian) system is employed for establishing the contribution of individual elements to the strain energy (U). The Eulerian displacements are obtained by means of element-based local vectors, where the vectors representing the principal axes of bending follow the deformed configuration of the element and are continuously updated to a position normal to the element chord. The nonlinear solution procedure is formalized in terms of transformations between the Eulerian and the global systems, and expressions for geometric stiffness and transformation matrices are explicitly derived.

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Verification examples, utilising an elastic quartic formulation and employing the nonlinear analysis program ADAPTIC, are presented to demonstrate the accuracy and versatility of this method in the large displacement analysis of space frames.

2. INTRODUCTION

The structural design process has evolved over the last few decades to allow safer, yet more economic, structures. This has largely been due to an increased understanding of the fundamental principles governing the structural behaviour up to and beyond the ultimate limit state, including such effects as material and geometric nonlinearities. In the context of space frames, the study of geometric nonlinearity effects has always been complicated by the fact that the effect of finite rotations about fixed axes in three-dimensional space is sequence-dependent, and that moments corresponding to such rotations in expressions for work are not fixed in direction (Argyris et al. 1978).

The recent advent of powerful computers has prompted a surge of research activities aiming at establishing methods which, albeit numerically demanding, are capable of accurately modelling the large displacement behaviour of structures. Of these, the displacement-based finite element method has been most widely applied, mainly due to its accuracy, fully-established mathematical basis, and suitability for computer implementation. The finite element method is based on the general principle that structural equilibrium under applied loading is achieved at displacements which correspond to stationary total potential energy of the structure, as expressed by,

$$\text{Equilibrium} \quad \Leftrightarrow \quad \frac{\partial \Pi}{\partial U_j} = 0 \quad (\forall j) \quad (1.a)$$

$$\Pi = U + W \quad (1.b)$$

$$U_j = \text{nodal displacement at freedom } (j) \quad (1.c)$$

$$U = \text{strain energy} \quad (1.d)$$

$$W = \text{loss of potential energy of the load} \quad (1.e)$$

At each node, translational and rotational freedoms are needed to define possible deformation shapes of the structure. Since the effect of finite rotations is sequence-dependent, the order of application of the rotations must be assumed (Besseling 1977). Alternatively, commutative definitions, such as semi-tangential rotations (Argyris et al. 1978), or incremental application of rotations (Oran 1973) may be employed.

The strain energy (U) can be obtained once the structure deformation shape is established in terms of nodal freedoms. On the other hand, the expression for the loss of potential energy of the load (W) is sensitive to the type of applied moments and the particular definition of rotations employed. For example, moments corresponding to an incremental application of rotations are almost moments about fixed axes, while moments corresponding to sequential finite rotations are neither fixed in direction nor orthogonal. However, since realistic structural loading rarely includes applied moments, such considerations are only of academic interest.

Hereafter, a new general procedure is proposed for modelling the effects of large displacements on the response of space frames to conservative loading. The procedure utilises a convected (Eulerian) system for the determination of the element strain energy, and models the effect of large rotations incrementally. Verification analyses demonstrate the potential of the developed method in the accurate and efficient nonlinear analysis of space frames.

3. TECHNIQUE FORMALIZATION

In modelling realistic space frames, the finite element method requires a number of elements to be used, where the deformation within each element is obtained from displacements at nodal points. Apart from the fact that most realistic loading does not include applied moments, the inclusion of moment loads results in complexities in the expression for the loss of potential energy of the load (W), rendering it dependent on the type of moments and the particular definition of rotations. Assuming that the loading consists only of forces applied at the nodal positions, the total potential energy can be expressed as,

$$\Pi = \sum_e ({}^eU) - \sum_j \mathbf{P}_j \cdot \mathbf{U}_j \quad (2.a)$$

$${}^eU = \text{element strain energy} \quad (2.b)$$

$$P_j = \text{global applied force at freedom (j)} \quad (2.c)$$

$$U_j = \text{global nodal displacement of freedom (j)} \quad (2.d)$$

At equilibrium, (Π) is always stationary with respect to the nodal freedoms, hence,

$$\frac{\partial \Pi}{\partial U_j} = 0 \Rightarrow \sum_e \frac{\partial ({}^eU)}{\partial U_j} - P_j = 0 \Rightarrow \quad (3)$$

$$P_j = \sum_e ({}^e_{g f_j}) \quad (4)$$

where,

$${}^e_{g f_j} = \frac{\partial ({}^eU)}{\partial U_j} \quad (5)$$

According to equation (4), stationary total potential energy of the structure is achieved if the global applied forces P are identical to the resistance forces assembled from element contributions ${}^e_{g f}$. Whereas the loading vector P does not include any moment loads, terms of ${}^e_{g f}$ corresponding to rotational freedoms are not necessarily zero. As discussed later, these terms correspond to element global moments whose type depend on the adopted definition of rotations. However, such consideration does not affect the finite element solution, since the total potential energy is stationary relative to all possible nodal displacements and orientations.

4. NATURE OF GLOBAL ROTATIONS

An incremental approach is adopted for the application of global rotations; an essential requirement for the development of a plastic hinge formulation (Izzuddin 1991). Oran (1973) used a first order rotation matrix to describe the transformation of vector a due to an increment of global rotations (α, β, γ) about the three global axes (X, Y, Z):

$$a'_i = \sum_{j=1}^3 r_{i,j} T_{i,j} a_j \quad (6.a)$$

with,

$${}^rT = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \quad (6.b)$$

However, this relationship applies to very small increments, since the orthogonality property of transformed unit vectors is satisfied only to the first order in rotations.

In the current work, a higher order transformation is employed. The derivation is obtained by applying the resultant rotation increment $|r|$ about the resultant rotation axis r , as shown in figure (1). The ensuing transformation is expressed as:

$$a'_i = \sum_{j=1}^3 {}^rT_{i,j} a_j \quad (7)$$

where,

$${}^rT = \begin{bmatrix} 1 - \frac{(\beta^2 + \gamma^2)}{2} & -\gamma + \frac{\alpha\beta}{2} & \beta + \frac{\alpha\gamma}{2} \\ \gamma + \frac{\alpha\beta}{2} & 1 - \frac{(\alpha^2 + \gamma^2)}{2} & -\alpha + \frac{\beta\gamma}{2} \\ -\beta + \frac{\alpha\gamma}{2} & \alpha + \frac{\beta\gamma}{2} & 1 - \frac{(\alpha^2 + \beta^2)}{2} \end{bmatrix} \quad (8)$$

The use of such a transformation reduces the amount of spurious lengthening of vectors upon incremental rotation. It also preserves to a greater extent the orthogonality property of unit vectors, as may be shown by the product of the various rows and columns of rT in equation (8).

5. NATURE OF GLOBAL MOMENTS

As discussed in section 3 above, whereas the loading vector P is assumed not to include any applied moments, the global element resistance vector ${}^e_g f$ contains non-zero moment terms. Although the nature of these moments does not affect the finite element solution in as much as displacements are concerned, the investigation of the effect of the definition of rotation adopted herein on these moments is found to be extremely instructive.

In the virtual work expression, the moments corresponding to the global rotation increments of the previous section are not fixed in direction. This can be demonstrated by equating the virtual work of these moments to that of moments about fixed axes, i.e:

$$\sum_{i=1}^3 f_{m_i} \delta^f r_i = \sum_{j=1}^3 m_j \delta r_j \quad (9)$$

where, $\delta^f r$ is the infinitesimal increment of rotations about fixed axes, and δr is the infinitesimal increment of rotations as defined in section 4.

The relationship between δr and $\delta^f r$ can be obtained by establishing the equivalence of their effects on the rotated vector $\mathbf{a}'$ as follows:

$$\delta \mathbf{a}'_i = \sum_{j=1}^3 \delta_r T_{i,j} \mathbf{a}_j \equiv \sum_{k=1}^3 \delta^f_r T_{i,k} \mathbf{a}'_k \Rightarrow \quad (10)$$

$$\sum_{j=1}^3 \delta_r T_{i,j} \mathbf{a}_j \equiv \sum_{k=1}^3 \delta^f_r T_{i,k} \sum_{j=1}^3 r T_{k,j} \mathbf{a}_j = \sum_{j=1}^3 \sum_{k=1}^3 \delta^f_r T_{i,k} r T_{k,j} \mathbf{a}_j \Rightarrow \quad (11)$$

$$\delta_r T_{i,j} \equiv \sum_{k=1}^3 \delta^f_r T_{i,k} r T_{k,j} \quad (12)$$

These relationships represents a set of 9 equations with only 3 unknowns ($\delta\alpha, \delta\beta, \delta\gamma$). Thus, a unique solution exists only for a transformation matrix r^T preserving exactly the orthogonality property of unit vectors. Since r^T in equation (8) is accurate to the second order in rotation, the solution of equation (10) can only be obtained to the first order.

From equation (8) it can be established that:

$$\delta_r T = \begin{bmatrix} -\beta \delta\beta - \gamma \delta\gamma & -\delta\gamma + \frac{\alpha \delta\beta + \beta \delta\alpha}{2} & \delta\beta + \frac{\alpha \delta\gamma + \gamma \delta\alpha}{2} \\ \delta\gamma + \frac{\alpha \delta\beta + \beta \delta\alpha}{2} & -\alpha \delta\alpha - \gamma \delta\gamma & -\delta\alpha + \frac{\beta \delta\gamma + \gamma \delta\beta}{2} \\ -\delta\beta + \frac{\alpha \delta\gamma + \gamma \delta\alpha}{2} & \delta\alpha + \frac{\beta \delta\gamma + \gamma \delta\beta}{2} & -\alpha \delta\alpha - \beta \delta\beta \end{bmatrix} \quad (13)$$

and to the first order,

$${}_{\mathbf{r}}\mathbf{T} = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \quad (14)$$

while the infinitesimal transformation due to rotations about fixed axes is given by:

$$\delta {}_{\mathbf{r}}^{\mathbf{f}}\mathbf{T} = \begin{bmatrix} 0 & -\delta\gamma^{\mathbf{f}} & \delta\beta^{\mathbf{f}} \\ \delta\gamma^{\mathbf{f}} & 0 & -\delta\alpha^{\mathbf{f}} \\ -\delta\beta^{\mathbf{f}} & \delta\alpha^{\mathbf{f}} & 0 \end{bmatrix} \quad (15)$$

Using equations (13) to (15), it can be demonstrated that the first order solution to equation (12) is:

$$\begin{Bmatrix} \delta\alpha \\ \delta\beta \\ \delta\gamma \end{Bmatrix} = \begin{bmatrix} 1 & \frac{\gamma}{2} & -\frac{\beta}{2} \\ -\frac{\gamma}{2} & 1 & \frac{\alpha}{2} \\ \frac{\beta}{2} & -\frac{\alpha}{2} & 1 \end{bmatrix} \begin{Bmatrix} \delta\alpha^{\mathbf{f}} \\ \delta\beta^{\mathbf{f}} \\ \delta\gamma^{\mathbf{f}} \end{Bmatrix} \quad (16)$$

or,

$$\delta \mathbf{r}_j = \sum_{i=1}^3 {}_{\mathbf{m}}\mathbf{T}_{i,j} \delta {}_{\mathbf{r}}^{\mathbf{f}}\mathbf{r}_i \quad (17)$$

where,

$${}_{\mathbf{m}}\mathbf{T} = \begin{bmatrix} 1 & -\frac{\gamma}{2} & \frac{\beta}{2} \\ \frac{\gamma}{2} & 1 & -\frac{\alpha}{2} \\ -\frac{\beta}{2} & \frac{\alpha}{2} & 1 \end{bmatrix} \quad (18)$$

The combination of (9) and (18) demonstrates that the global moments corresponding to the current definition of rotations are related to moments about fixed axes by the expression:

$${}_{\mathbf{f}}\mathbf{m}_i = \sum_{j=1}^3 {}_{\mathbf{m}}\mathbf{T}_{i,j} \mathbf{m}_j \quad (19)$$

It is therefore evident that the current global moments are not moments about fixed axes, but they are rather defined in a system rotated by **half** the increment of rotations (α, β, γ) , as shown in figure

(2). This corresponds to the semi-tangential moment definition (Argyris et al. 1978), only it is applied in the context of an incremental approach. However, as long as the increments of rotation (α, β, γ) are small, $\mathbf{m}^T$ is approximately an identity matrix, and the global moments can be assumed to be about fixed axes.

6. THE EULERIAN SYSTEM

The Eulerian (convected) system is a local reference system which follows the element chord during deformation, as shown in figure (3). This system is convenient for the definition of the element strain energy (${}^e U$) in the presence of large nodal displacements, since it isolates the components of displacement resulting in element strains from displacements associated with stress-free rigid body motion.

We first define vector ${}_g \mathbf{u}$ as the *incremental* global element displacements (figure 3), and vector ${}_c \mathbf{u}$ as the *total* local element displacements (figure 4), i.e.:

$${}_g \mathbf{u} = \langle u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rangle^T \quad (20)$$

$${}_c \mathbf{u} = \langle \theta_{1y}, \theta_{1z}, \theta_{2y}, \theta_{2z}, \Delta, \theta_T \rangle^T \quad (21)$$

In order to obtain the local Eulerian displacements ${}_c \mathbf{u}$, the effect of the incremental global displacements and rotations ${}_g \mathbf{u}$ on a set of chosen element vectors is quantified. These vectors represent the element chord and the two principal axes of bending at the two ends, as shown in figure (5). This approach is different from that employed in previous investigations (Oran 1973, Meek and Loganathan 1989) in that element-based rather than nodal triad vectors are used to quantify the local displacements. This is necessary for the inclusion of large local rotations, which usually exist in plastic hinge analysis (Izzuddin 1991).

At the last equilibrium configuration, the two vector sets, $\begin{pmatrix} 1_c^0 \\ y^0 \end{pmatrix}$ and $\begin{pmatrix} 2_c^0 \\ z^0 \end{pmatrix}$, representing the principal axes at the element ends, are normal to the chord vector x_c^0 , but not necessarily identical; the latter consideration depends on the cumulative twist of the element.

The vectors of principal axes are modified according to the increment of global rotations at the two ends, while the chord vector is only affected by translational displacements. Hence,

$$\mathbf{x}^c = \left\langle \frac{X_E^c}{L^c}, \frac{Y_E^c}{L^c}, \frac{Z_E^c}{L^c} \right\rangle^T \quad (22.a)$$

$${}^1\mathbf{y}^c_i = \sum_{j=1}^3 {}^1\mathbf{T}_{i,j} {}^1\mathbf{y}^c_j \quad (22.b)$$

$${}^1\mathbf{z}^c_i = \sum_{j=1}^3 {}^1\mathbf{T}_{i,j} {}^1\mathbf{z}^c_j \quad (22.c)$$

$${}^2\mathbf{y}^c_i = \sum_{j=1}^3 {}^2\mathbf{T}_{i,j} {}^2\mathbf{y}^c_j \quad (22.d)$$

$${}^2\mathbf{z}^c_i = \sum_{j=1}^3 {}^2\mathbf{T}_{i,j} {}^2\mathbf{z}^c_j \quad (22.e)$$

where,

$$X_E^c = X_E^o + u_2 - u_1 \quad (23.a)$$

$$Y_E^c = Y_E^o + v_2 - v_1 \quad (23.b)$$

$$Z_E^c = Z_E^o + w_2 - w_1 \quad (23.c)$$

$$L^c = \sqrt{(X_E^c)^2 + (Y_E^c)^2 + (Z_E^c)^2} \quad (23.d)$$

and,

$${}^1\mathbf{T}_r = {}^1\mathbf{T}(\alpha_1, \beta_1, \gamma_1) \quad (24.a)$$

$${}^2\mathbf{T}_r = {}^2\mathbf{T}(\alpha_2, \beta_2, \gamma_2) \quad (24.b)$$

with ${}^r\mathbf{T}(\alpha, \beta, \gamma)$ defined in (8).

The inclination of the current principal axes relative to the chord axis determines the increment of relative element rotations. However, for the calculation of the increment of twist, a fictitious vector ${}^{21}\mathbf{y}^c$ is defined as the transformation of the principal y-axis at end (1) due to rotation increments at end (2). Assuming small incremental values, the increment of chord displacements δ_c^u can be established according to the following relations:

$$\delta_c^u = \left\langle \delta\theta_{1y}, \delta\theta_{1z}, \delta\theta_{2y}, \delta\theta_{2z}, \delta\Delta, \delta\theta_T \right\rangle^T \quad (25.a)$$

$$\delta\theta_{1y} = -\mathbf{x}^c \bullet {}^1\mathbf{y}^c \quad (25.b)$$

$$\delta\theta_{1z} = -x^c \bullet \frac{1}{z^c} \quad (25.c)$$

$$\delta\theta_{2y} = -x^c \bullet \frac{2}{y^c} \quad (25.d)$$

$$\delta\theta_{2z} = -x^c \bullet \frac{2}{z^c} \quad (25.e)$$

$$\delta\Delta = L^c - L^o \quad (25.f)$$

$$\delta\theta_T = \frac{1}{z^c} \bullet \frac{21}{y^c} \quad (25.g)$$

where,

$$L^o = \sqrt{(X_E^o)^2 + (Y_E^o)^2 + (Z_E^o)^2} \quad (26)$$

and,

$$\frac{21}{y^c} = \sum_{j=1}^3 \frac{2}{r} T_{i,j} \frac{1}{y^c} \quad (27)$$

The current chord displacements c^u are obtained by updating the displacements of the last configuration:

$$c^u = c^o + \delta c^u \quad (28)$$

Once equilibrium is achieved, the principal axes are reset to a position normal to the current chord axis, so that the above equations can be applied for the subsequent incremental step. This is performed according to the following procedure:

$$x^o = x^c \quad (29.a)$$

$$\frac{1}{y^o} = \frac{\frac{1}{z^c} \times x^c}{\left| \frac{1}{z^c} \times x^c \right|} \quad (29.b)$$

$$\frac{1}{z^o} = x^c \times \frac{1}{y^o} \quad \left(\frac{1}{y^o} \text{ after resetting} \right) \quad (29.c)$$

$$\frac{2}{y^o} = \frac{\frac{2}{z^c} \times x^c}{\left| \frac{2}{z^c} \times x^c \right|} \quad (29.d)$$

$$\frac{2}{z^o} = x^c \times \frac{2}{y^o} \quad \left(\frac{2}{y^o} \text{ after resetting} \right) \quad (29.e)$$

7. ELEMENT STRAIN ENERGY

The solution of the equations of equilibrium, represented by equation (4), requires the definition of element global forces ${}^e_g f$ which are the first derivatives of the element strain energy (${}^e U$) with respect to the global displacements U , as given by equation (5). However, since (${}^e U$) is more conveniently expressed in terms of local displacements ${}_c u$, chain differentiation rules are used to obtain ${}^e_g f$. An equivalent form of equation (5), considering only the element global freedoms ${}_g u$, is hence given by:

$${}^e_g f_j = \frac{\partial({}^e U)}{\partial {}_g u_j} = \sum_{i=1}^6 \frac{\partial {}_c u_i}{\partial {}_g u_j} \frac{\partial({}^e U)}{\partial {}_c u_i} \Rightarrow \quad (30)$$

$${}^e_g f_j = \sum_{i=1}^6 T_{j,i} {}_c f_i \quad (31)$$

where,

$$T_{j,i} = \frac{\partial {}_c u_i}{\partial {}_g u_j} \quad (32)$$

and,

$${}_c f_i = \frac{\partial({}^e U)}{\partial {}_c u_i} \quad (33)$$

Vector ${}_c f$ represents local element forces which depend on the element type, as discussed in section 8. On the other hand, matrix T is employed for the transformation of local to global element forces, and its terms can be explicitly obtained from equations (20) to (29), as given in Appendix (A.1).

8. LOCAL ELEMENT FORCES

The local element forces ${}_c f$, referred to in section 7, are first derivatives of the strain energy (${}^e U$) with respect to local displacements ${}_c u$, and depend on the element type.

For ordinary beam elements employing cubic and linear shape functions for the transverse and axial displacements, respectively, c^f is given by the following relationship:

$$c^f_i = \sum_{j=1}^6 c^k_{i,j} c^u_j \quad (34)$$

where,

$$c^k = \frac{1}{L} \begin{bmatrix} 4EI_y & 0 & 2EI_y & 0 & 0 & 0 \\ 0 & 4EI_z & 0 & 2EI_z & 0 & 0 \\ 2EI_y & 0 & 4EI_y & 0 & 0 & 0 \\ 0 & 2EI_z & 0 & 4EI_z & 0 & 0 \\ 0 & 0 & 0 & 0 & EA & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ \end{bmatrix} \quad (35)$$

In the current work, a quartic formulation, discussed in detail in Ref. (Izzuddin 1991), is used for demonstrating the applicability of the developed procedures to the large displacement analysis of space frames. The quartic formulation has eight degrees of freedom in the local system, as shown in figure (6), and provides the capability for modelling initial imperfections. The inclusion in this formulation of the additional mid-side transverse freedoms, accompanied by accounting for the effect of bowing on the centroidal axial strain, result in significant improvement in accuracy (Izzuddin 1991) over earlier beam-column formulations (Meek and Loganathan 1989, Jennings 1968).

9. SOLUTION PROCEDURE

The solution of the nonlinear system, represented by equation (4), requires the use of incremental iterative strategies, such as the family of Newton-Raphson procedures. For this purpose, an element global tangent stiffness matrix g^k must be derived, with assembly procedures used to form the global tangent stiffness matrix of the overall structure.

The element global tangent stiffness g^k can be obtained from equation (31) using chain differentiation properties, as given below:

$$g_{i,j}^k = \frac{\partial f_i}{\partial u_j} = \sum_{k=1}^6 \left(T_{i,k} \frac{\partial c_k}{\partial u_j} + \frac{\partial T_{i,k}}{\partial u_j} c_k \right) \Rightarrow \quad (36)$$

$$g_{i,j}^k = \sum_{k=1}^6 \left(T_{i,k} \left\{ \sum_{m=1}^6 \frac{\partial c_k}{\partial u_m} \frac{\partial u_m}{\partial u_j} \right\} + \frac{\partial T_{i,k}}{\partial u_j} c_k \right) \quad (37)$$

but,

$$\frac{\partial c_k}{\partial u_m} = c_{k,m} \quad (38.a)$$

$$\frac{\partial u_m}{\partial u_j} = T_{j,m} \quad (38.b)$$

$$\frac{\partial T_{i,k}}{\partial u_j} = \frac{\partial^2 u_k}{\partial u_i \partial u_j} = s_{i,j,k} G_{i,j,k} \quad (38.c)$$

hence,

$$g_{i,j}^k = \sum_{k=1}^6 \left(\left\{ \sum_{m=1}^6 T_{i,k} c_{k,m} T_{j,m} \right\} + s_{i,j,k} G_{i,j,k} c_k \right) \quad (39)$$

The stiffness contribution $\left(s_{i,j,k} G_{i,j,k} c_k \right)$ is referred to as *geometric stiffness*, since it reflects the effect of a change in geometry on the global forces. The geometric matrix $s \mathbf{G}$ is a 12 x 12 x 6 matrix of second derivatives of local with respect to global displacements, the explicit terms of which can be obtained by differentiating once the terms of matrix T given in Appendix I (Izzuddin 1991).

10. EXAMPLES

The previously described procedures have been implemented in ADAPTIC, a general purpose computer program for the nonlinear static and dynamic analysis of steel, reinforced concrete, and composite frames (Izzuddin and Elnashai 1989). Three comparative examples are performed using ADAPTIC, and presented herein to demonstrate the accuracy of the developed procedures.

10.1 Circular bend

The 45° circular bend depicted in figure (7) has been extensively used for the verification of finite element formulations (Bathe and Bolourchi 1979, Surana and Sorem 1989). The bend is analysed here using 1 and 2 imperfect quartic elements, and applying the load in 15 equal increments. With 2 elements, the quartic formulation gives identical results to those in Ref. (Surana and Sorem 1989), as shown in figure (8). Whereas 1 quartic element still provides a very good comparison even though the bend curvature corresponds to high imperfection levels, thus compromising the assumption of small deformation relative to the element chord.

The bend is re-analysed using 2 imperfect quartic elements, but applying the load in 5 increments only. The comparison in figure (9) demonstrates that the adopted 3D incremental approach is almost insensitive to the size of the load increment applied.

10.2 Lateral torsional buckling

In this work, lateral torsional instability is not given a high priority due to the complexities associated with extra warping freedoms and the modelling of warping in the presence of section plasticity. This is the main reason for neglecting the flexure-torsion coupling in the derivation of the quartic formulation in the Eulerian system. However, in the absence of warping, such coupling can still be accounted for if a sufficient number of elements is used per member, since the effect of nodal displacements and rotations on geometry is accurately accounted for.

To demonstrate this, a thin rectangular cantilever strip is subjected to a semi-tangential moment (M) at its tip, and assumed to have an initial out-of-straightness of ($L/1000$) to initiate lateral buckling, as shown in figure (10). The load-deflection response given by 4 quartic elements in figure (11) provides a good prediction of the theoretical buckling moment (M_b) (Argyris et al. 1978). The buckling characteristics are still exhibited when using 1 quartic element, but the predicted response is over-stiff as expected, since the flexure-torsion coupling is neglected in the Eulerian system.

10.3 Elastic dome

The framed dome of figure (12) was analysed by Remseth (1979) who employed a Lagrangian beam-column formulation neglecting large global rotations. The same structure was later considered by Shi and Atluri (1988) who applied transformations which become erroneous when large nodal rotations are involved.

The static response to a concentrated load at the crown point is shown in figure (13), where excellent agreement is demonstrated between the prediction of ADAPTIC employing one elastic quartic element per member and the results based on the formulation of Kondoh et al (Kondoh et al. 1986). The disagreement with the results of Shi and Atluri, and more significantly with those of Remseth, can be attributed to the point mentioned above regarding their consideration of large global rotations.

11. CONCLUSION

A new procedure was developed to model the effects of large displacements on the behaviour of space frames. The development is entirely based on the principle of stationary total potential energy, with appropriate choice of reference systems allowing the procedure formalization in terms of transformations between the Eulerian and global systems. It was demonstrated that different modelling approaches of large rotations would lead to the same solution, provided that there are no applied moments.

Although an incremental approach was used to model the effect of large rotations, an improved rotational transformation matrix was adopted in order to reduce spurious lengthening of vectors upon rotation. Moreover, the calculation of rotations in the Eulerian system was performed incrementally using element-based vectors. Consequently, large relative rotations, which usually exist in plastic hinge analysis, are readily accommodated.

Examples performed using the nonlinear analysis program ADAPTIC demonstrated the accuracy of the proposed procedure. It was also shown that the incremental strategy is almost insensitive to the size of the load step.

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REFERENCES

- Argyris J.H., Dunne P.C. and Scharpf D.W., 1978. "On large displacement small strain analysis of structures with rotational degrees of freedom", *Comp. Meth. Appl. Mech. Eng.*, (Vol. 14, pp. 401-451) continued on (Vol. 15, pp. 99-135).
- Bathe K.J. and Bolourchi S., 1979. "Large displacement analysis of three-dimensional beam structures", *Int. J. Num. Meth. Eng.*, Vol. 14, pp. 961-986.
- Besseling J.F., 1977. "Derivatives of deformation parameters for bar elements and their use in buckling and postbuckling analysis", *Comp. Meth. Appl. Mech. Eng.*, Vol. 12, pp. 97-124.
- Izzuddin B.A. and Elnashai A.S., 1989. "ADAPTIC: A Program for the Adaptive Dynamic Analysis of Space Frames", Rep. No. ESEE-89/7, Imperial College, London.
- Izzuddin B.A., 1991. "Nonlinear dynamic analysis of framed structures", Thesis submitted for the degree of Doctor of Philosophy in the University of London, Department of Civil Engineering, Imperial College, London.
- Jennings A., 1968. "Frame analysis including change of geometry", *J. Struct. Div., ASCE*, Vol. 94, No. ST3, pp. 627-644.
- Kondoh K., Tanaka K. and Atluri S.N., 1986. "An explicit expression for the tangent-stiffness of a finitely deformed 3-D beam and its use in the analysis of space frames", *Comp. Struct.*, Vol. 24, No. 2, pp. 253-271.

- Meek J.L. and Loganathan S., 1989. "Geometrically non-linear behaviour of space frame structures", *Comp. Struct.*, Vol. 31, No. 1, pp. 35-45.
- Oran C., 1973. "Tangent stiffness in space frames", *J. Struct. Div., ASCE*, Vol. 99, No. ST6, pp. 987-1001.
- Remseth S.N., 1979. "Nonlinear static and dynamic analysis of framed structures", *Comp. Struct.*, Vol. 10, pp. 879-897.
- Shi G. and Atluri S.N., 1988. "Elasto-plastic large deformation analysis of space-frames: A plastic-hinge and stress-based derivation of tangent stiffnesses", *Int. J. Num. Meth. Eng.*, Vol. 26, pp. 589-615.
- Surana K.S. and Soreem R.M., 1989. "Geometrically non-linear formulation for three dimensional curved beam elements with large rotations", *Int. J. Num. Meth. Eng.*, Vol. 28, pp. 43-73.

APPENDIX A

A.1 Derivation of matrix **T**

The transformation matrix **T** referred to in equation (31) is a matrix of first derivatives of local with respect to global displacements. The 12x6 terms of **T** can be established explicitly using equations (20)-(28), as shown below.

$$\mathbf{c}^u = \langle \theta_{1y}, \theta_{1z}, \theta_{2y}, \theta_{2z}, \Delta, \theta_T \rangle^T \quad (40)$$

$$\mathbf{g}^u = \langle u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rangle^T \quad (41)$$

$${}^1_t\mathbf{A} = \{1, 2, 3\} = \text{numbers of translational global freedoms at end(1)} \quad (42.a)$$

$${}^1_r\mathbf{A} = \{4, 5, 6\} = \text{numbers of rotational global freedoms at end(1)} \quad (42.b)$$

$${}^2_t\mathbf{A} = \{7, 8, 9\} = \text{numbers of translational global freedoms at end(2)} \quad (42.c)$$

$${}^2_r\mathbf{A} = \{10, 11, 12\} = \text{numbers of rotational global freedoms at end(2)} \quad (42.d)$$

$${}_t\mathbf{A} = {}^1_t\mathbf{A} \cup {}^2_t\mathbf{A} = \text{numbers of translational global freedoms} \quad (42.e)$$

$${}_r\mathbf{A} = {}^1_r\mathbf{A} \cup {}^2_r\mathbf{A} = \text{numbers of rotational global freedoms} \quad (42.f)$$

$${}_a\mathbf{A} = {}_t\mathbf{A} \cup {}_r\mathbf{A} = \text{numbers of all global freedoms} \quad (42.g)$$

$$T_{i,1} = \frac{\partial \theta_{1y}}{\partial g_{u_i}} \quad (43.a)$$

$$T_{i,1} = - \frac{\partial x^c}{\partial g_{u_i}} \bullet \frac{1}{y^c} \quad i \in {}_t\mathbf{A} \quad (43.b)$$

$$T_{i,1} = - x^c \bullet \left(\frac{\partial {}^1_r T}{\partial g_{u_i}} \frac{1}{y^c} \right) \quad i \in {}^1_r\mathbf{A} \quad (43.c)$$

$$T_{i,1} = 0 \quad i \in {}^2_r\mathbf{A} \quad (43.d)$$

$$T_{i,2} = \frac{\partial \theta_{1z}}{\partial g u_i} \quad (44.a)$$

$$T_{i,2} = -\frac{\partial_x c^c}{\partial g u_i} \bullet \frac{1}{z} c^c \quad i \in {}_t A \quad (44.b)$$

$$T_{i,2} = -x c^c \bullet \left(\frac{\partial_r^1 T}{\partial g u_i} \frac{1}{z} c^o \right) \quad i \in {}_r^1 A \quad (44.c)$$

$$T_{i,2} = 0 \quad i \in {}_r^2 A \quad (44.d)$$

$$T_{i,3} = \frac{\partial \theta_{2y}}{\partial g u_i} \quad (45.a)$$

$$T_{i,3} = -\frac{\partial_x c^c}{\partial g u_i} \bullet \frac{2}{y} c^c \quad i \in {}_t A \quad (45.b)$$

$$T_{i,3} = 0 \quad i \in {}_r^1 A \quad (45.c)$$

$$T_{i,3} = -x c^c \bullet \left(\frac{\partial_r^2 T}{\partial g u_i} \frac{2}{y} c^o \right) \quad i \in {}_r^2 A \quad (45.d)$$

$$T_{i,4} = \frac{\partial \theta_{2z}}{\partial g u_i} \quad (46.a)$$

$$T_{i,4} = -\frac{\partial_x c^c}{\partial g u_i} \bullet \frac{2}{z} c^c \quad i \in {}_t A \quad (46.b)$$

$$T_{i,4} = 0 \quad i \in {}_r^1 A \quad (46.c)$$

$$T_{i,4} = -x c^c \bullet \left(\frac{\partial_r^2 T}{\partial g u_i} \frac{2}{z} c^o \right) \quad i \in {}_r^2 A \quad (46.d)$$

$$T_{i,5} = \frac{\partial \Delta}{\partial g u_i} \quad (47.a)$$

$$T_{i,5} = \frac{\partial L_c}{\partial g u_i} \quad i \in {}_t A \quad (47.b)$$

$$T_{i,5} = 0 \quad i \in {}_r A \quad (47.c)$$

$$T_{i,6} = \frac{\partial \theta_T}{\partial g u_i} \quad (48.a)$$

$$T_{i,6} = 0 \quad i \in {}_tA \quad (48.b)$$

$$T_{i,6} = \left(\frac{\partial_r^1 T}{\partial g u_i} {}^1_z c^o \right) \bullet {}^{21}_y c \quad i \in {}^1_r A \quad (48.c)$$

$$T_{i,6} = {}^1_z c^c \bullet \left(\frac{\partial_r^2 T}{\partial g u_i} {}^1_y c^o \right) \quad i \in {}^2_r A \quad (48.d)$$

where,

$$\frac{\partial_x c_k^c}{\partial g u_i} = \frac{\left(x^c c_i \ x^c c_k - I_{i,k} \right)}{L^c} \quad i \in {}^1_t A \quad (49.a)$$

$$\frac{\partial_x c_k^c}{\partial g u_i} = - \frac{\partial_x c_k^c}{\partial g u_{i-6}} \quad i \in {}^2_t A \quad (49.b)$$

$$I = 3 \times 3 \text{ identity matrix} \quad (49.c)$$

$$\frac{\partial_r T}{\partial \alpha} = \begin{bmatrix} 0 & \frac{\beta}{2} & \frac{\gamma}{2} \\ \frac{\beta}{2} & -\alpha & -1 \\ \frac{\gamma}{2} & 1 & -\alpha \end{bmatrix} \quad (50.a)$$

$$\frac{\partial_r T}{\partial \beta} = \begin{bmatrix} -\beta & \frac{\alpha}{2} & 1 \\ \frac{\alpha}{2} & 0 & \frac{\gamma}{2} \\ -1 & \frac{\gamma}{2} & -\beta \end{bmatrix} \quad (50.b)$$

$$\frac{\partial_r T}{\partial \gamma} = \begin{bmatrix} -\gamma & -1 & \frac{\alpha}{2} \\ 1 & -\gamma & \frac{\beta}{2} \\ \frac{\alpha}{2} & \frac{\beta}{2} & 0 \end{bmatrix} \quad (50.c)$$

and,

$$\frac{\partial L^c}{\partial g^u_i} = -{}_x c^c \quad i \in {}^1_t A \tag{51.a}$$

$$\frac{\partial L^c}{\partial g^u_i} = -\frac{\partial L^c}{\partial g^u_{i-6}} \quad i \in {}^2_t A \tag{51.b}$$

NOTATION

- Generic symbols of matrices and vectors are represented by bold font-type with left side subscripts or superscripts (e.g. $\mathbf{s}^{\mathbf{G}}$, $\mathbf{q}^{\mathbf{a_u}}$). This rule also applies to three-dimensional matrices.
- Subscripts and superscripts to the right side of the generic symbol indicate the term of the vector or matrix under consideration (e.g. $\mathbf{s}^{\mathbf{G}_{i,j,k}}$, $\mathbf{q}^{\mathbf{a_u}_i}$).

Operators

- $\mathbf{c}$: right-side superscript, denotes current values during an incremental step.
- $\mathbf{o}$: right-side superscript, denotes initial values during an incremental step.
- $\mathbf{T}$: right-side superscript, transpose sign.
- δ : incremental operator for variables, vectors and matrices.
- ∂ : partial differentiation.
- $\sum_i$: summation over range variable (i).
- $[\]$: encloses terms of a matrix.
- $\{ \}$: encloses terms of a row vector.
- $|\mathbf{a}|$: magnitude of vector $\mathbf{a}$.
- $\bullet$: vector dot product.
- $\times$: vector cross product.
- $\forall$: for all.

Symbols

- $\mathbf{a}$: vector in 3D space.
- $\mathbf{a}'$: vector in 3D space after rotation.
- $\mathbf{A}$: cross-sectional area.
- $\mathbf{c}_x$: vector of direction cosines of local x-axis of 3D element.
- $\mathbf{l}_y$: vector of direction cosines of local y-axis at end(1).
- $\mathbf{l}_z$: vector of direction cosines of local z-axis at end(1).

- $\begin{matrix} 2 \\ c \\ y \end{matrix}$: vector of direction cosines of local y-axis at end(2).
 $\begin{matrix} 2 \\ c \\ z \end{matrix}$: vector of direction cosines of local z-axis at end(2).
 $\begin{matrix} 2 \\ 1 \\ c \\ y \end{matrix}$: fictitious vector of direction cosines of local y-axis at end(1) due to rotations at end(2).
 E : Young's modulus of elasticity.
 $\begin{matrix} f \\ c \end{matrix}$: element basic local forces
 $\langle M_{1y}, M_{1z}, M_{2y}, M_{2z}, F, M_T \rangle$.
 $\begin{matrix} f \\ g \end{matrix}$: element global forces.
 G : shear modulus.
 $\begin{matrix} G \\ s \end{matrix}$: element geometric stiffness matrix.
 I_y : second moment of inertia in the local y direction.
 I_z : second moment of inertia in the local z direction.
 I : identity matrix.
 J : St. Venant's torsion constant.
 $\begin{matrix} k \\ c \end{matrix}$: element local tangent stiffness matrix.
 $\begin{matrix} k \\ g \end{matrix}$: element global tangent stiffness matrix.
 L : element length before deformation.
 P : structure loading vector.
 r : rotational vector in 3D space
 $\langle \alpha, \beta, \gamma \rangle$.
 T : matrix for transformation of global to local displacements.
 $\begin{matrix} T \\ r \end{matrix}$: rotational transformation matrix in 3D space
 $\begin{matrix} 1 \\ T \\ r \end{matrix}$, transformation due to rotations at end(1) of element
 $\begin{matrix} 2 \\ T \\ r \end{matrix}$, transformation due to rotations at end(2) of element.
 $\begin{matrix} u \\ c \end{matrix}$: element basic local displacements
 $\langle \theta_{1y}, \theta_{1z}, \theta_{2y}, \theta_{2z}, \Delta, \theta_T \rangle$.
 $\begin{matrix} u \\ g \end{matrix}$: incremental element global displacements
 $\langle u_1, v_1, w_1, \alpha_1, \beta_1, \gamma_1, u_2, v_2, w_2, \alpha_2, \beta_2, \gamma_2 \rangle$.
 U : strain energy.
 U : global displacements of structure.

W : loss of potential energy of the load.
 X_E : element global X-axis coordinate.
 Y_E : element global Y-axis coordinate.
 Z_E : element global Z-axis coordinate.
 Π : total potential energy.

Figure 1. Application of global rotations

Figure 2. Orientation of global moments relative to fixed global axes

Figure 3. The Eulerian (convected) system in relation to a global reference system

Figure 4. Local Eulerian displacements

Figure 5. Local element vectors for the current and previous configurations

Figure 6. Local freedoms of the quartic formulation

Figure 7. Geometry and loading of circular bend

Figure 8. Response prediction of circular bend

Figure 9. Effect of number of load increments on response prediction

Figure 10. Rectangular strip subjected to semi-tangential moment

Figure 11. Load-displacement response of 3D cantilever strip

Figure 12. Geometric configuration of elastic dome

Figure 13. Static response of elastic dome

$$\begin{aligned}
 E &= 10 \times 10^6 \text{ lb} / \text{in}^2 \\
 \nu &= 0.3 \\
 EI_y &= EI_z = 8.333 \times 10^5 \text{ lb} \cdot \text{in}^2 \\
 GJ &= 5.417 \times 10^5 \text{ lb} \cdot \text{in}^2
 \end{aligned}$$

$$k = \frac{P}{\left(\frac{EI}{R^2} \right)}$$